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Construction of Magic Squares Using the Knight's Move in Islamic Mathematics

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1. Introduction

One of the most impressive if not most original achievements in Islamic mathematics was the development of general methods for constructing magic squares. A magic square of order n is a square with n cells on its side, thus n^2 cells on the whole, in which different natural numbers are arranged in such a way that the sum of each line, column and main diagonal is the same (Fig. 1 and 2). Such are the properties of *simple* magic squares. As a rule, the n^2 first natural numbers are actually written in, which means that the constant sum amounts to $\frac{1}{2}n(n^2 + 1)$, the n-th part of their sum. If the squares left when the borders are successively removed are themselves magic, the square is called *bordered* (Fig. 3). If every pair of broken diagonals (that is, two diagonals which lie on either side of, and parallel to, a main diagonal and together have n cells) shows the constant sum, the square is called *pandiagonal* (Fig. 4, where, for example, the sums $17 + 1 + 15 + 24 + 8$ and $2 + 10 + 13 + 16 + 24$ are also equal to the magic sum 65). Then there are *composite* squares: when the order n is a composite number, say $n = r \cdot s$ with $r, s \geq 3$, the main square can be divided into r^2 subsquares of order s; these subsquares, taken successively according to a magic arrangement for the order r, are then filled with sequences of s^2 consecutive numbers according to a magic arrangement for the order s, the result being a magic square in which each subsquare is also magic (Fig. 5, constructed according to the squares in Fig. 6 and 7, thus $r = 3$, $s = 4$).

Magic squares are usually divided into three categories according to order: *odd* when n is odd, that is, $n = 3, 5, 7, \ldots$, and generally $n = 2k+1$ with k natural; *evenly-even* if n is even and divisible by 4, thus $n = 4, 8, 12, \ldots, 4k$; and, finally, *oddly-even* if n is even but divisible by 2 only, whence $n = 6, 10, 14, \ldots, 4k+2$. There are general methods of construction, depending on type (simple, bordered, pandiagonal) and category. Those for *simple* squares may, however, not apply for the smallest orders $n = 3$ and $n = 4$, which are particular cases. Those for *bordered* squares suppose that $n \geq 5$. (Since no square of order 2 is possible with different numbers, no square of order 4 can be bordered.) Finally, those for odd-order *pandiagonal* squares are generally not directly applicable if n is divisible by 3, and there are no rules for constructing oddly-even squares since such squares do not exist.

Information about the beginning of Islamic research on magic squares is lacking. It may have been connected with the introduction of chess into Persia. Initially, the problem was purely mathematical: whence the ancient Arabic designation for magic squares

1	31	22	15	30	12
2	32	14	23	29	11
3	33	24	13	28	10
34	4	16	21	9	27
35	5	17	20	8	26
36	6	18	19	7	25

Figure 1

4	40	8	1
9	6	24	14
17	5	18	13
23	2	3	25

Figure 2

92	17	4	95	8	91	12	87	16	83
99	76	31	22	77	26	73	30	69	2
1	20	64	41	36	63	40	59	81	100
3	19	67	58	47	51	46	34	82	98
96	80	33	52	45	57	48	68	21	5
7	78	35	49	56	44	53	66	23	94
90	27	62	43	54	50	55	39	74	11
13	72	42	60	65	38	61	37	29	88
86	32	70	79	24	75	28	71	25	15
18	84	97	6	93	10	89	14	85	9

Figure 3

of *wafq al-a‘d ād,* that is, “harmonious disposition of the numbers”. Although we know that treatises were written in the ninth century, the earliest extant ones date from the tenth: one is by Abū’l-Wafā’ al-Būzjānī (940-997 or 998) and the other is a chapter in Book III of ‘Alī b. Aḥmad al-Anṭākī’s (d. 987) *Commentary on Nicomachos’ Arithmetic* (Sesiano 1998a and 2003). By that time, the science of magic squares was apparently well established: it was known how to construct bordered squares of any order as well as simple magic squares of small orders ($n \leq 6$), which were used for making composite

21	17	13	9	5
14	10	1	22	18
2	23	19	15	6
20	11	7	3	24
8	4	25	16	12

Figure 4

49	62	59	56	129	142	139	136	17	30	27	24
60	55	50	61	140	135	130	141	28	23	18	29
54	57	64	51	134	137	144	131	22	25	32	19
63	52	53	58	143	132	133	138	31	20	21	26
33	46	43	40	65	78	75	72	97	110	107	104
44	39	34	45	76	71	66	77	108	103	98	105
38	41	48	35	70	73	80	67	102	105	112	99
47	36	37	42	79	68	69	74	111	100	101	106
113	126	123	120	1	14	11	8	81	94	91	88
124	119	114	125	12	7	2	13	92	87	82	93
118	121	128	115	6	9	16	3	86	89	96	83
127	116	117	122	15	4	5	10	95	84	85	90

Figure 5

4	9	2
3	5	7
8	1	6

Figure 6

squares. (Although methods for simple magic squares are easier to *apply* than methods for bordered ones, the latter are easier to *discover*.) The 11th century saw the finding of several ways to construct simple magic squares, in any event for odd and evenly-even orders (see the two anonymous treatises, presumably from the first half of the 11th century, in Sesiano 1996a, 1996b); and the more difficult case of $n = 4k + 2$, which Ibn al-Haytham (c. 965-1040) could solve only with k even (Sesiano 1980), was settled by the beginning of the 12th century (Sesiano 1995), if not already in the second half of

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

Figure 7

the eleventh. At the same time, pandiagonal squares of evenly-even order were being constructed, and of odd order with n not divisible by 3. (Little attention seems to have been paid to the sum of the broken diagonals; these squares were considered of interest because the initial cell, that is, the place of 1, could vary within the square.) Treatises on magic squares were numerous in the 12th century, and later developments tended to be improvements on or simplifications of existing methods. From the 13th century onwards, magic squares were increasingly put to magic purposes.

The connexion with magic arose from the association of each of the twenty-eight Arabic letters with a number (the units, the tens, the hundreds and one thousand). Thus to a name or a sentence corresponded a determined numerical quantity: whence the idea of writing in, say, the first row the sequence of numbers equivalent to either the letters of the word or the words of the sentence, then completing the square so as to produce the same sum in each line. This, however, involved a completely different kind of construction, which depended upon the order n and the values of the n given quantities. The problem is mathematically not easy, and led in the 11th century to interesting constructions for the cases $n = 3$ to $n = 8$ (Sesiano 1996b). Since few people interested in magic and talismans had much taste for mathematics, most texts written for them merely depicted certain magic squares and mentioned their attributes; some did, however, keep the general theory alive, such as one, of uneven value, by the 17th-century Egyptian Muḥammad Shabrāmallisī. Among the sets of magic squares used for talismanic purposes, we also find simple magic squares of the first seven possible orders ($n = 3$ to $n = 9$) filled with the first natural numbers; each is associated with one of the seven heavenly bodies then known and was supposed to be endowed with the same virtues and defects as the corresponding planet.

The transmission of Islamic research on magic squares was uneven. Thus it was that Europe only received, in the late Middle Ages, two sets of squares associated with the planets in magic texts and without any indication as to their construction. (Because of these sources, in Europe such squares came to be called magic and also, until the 17th century, planetary.) No other Arabic text on magic squares reached Europe or, at any rate, appears to have been studied or used there. The extent of Islamic research thus remained unknown for quite a long time; indeed, a very long time, since it has only recently been assessed and its importance recognized. The East was more fortunate. As early as the twelfth century some methods of construction had reached India and China; and also Byzantium, as can be seen from the treatise on magic squares written around 1300 by Manuel Moschopoulos, which is also the first mediaeval treatise on magic squares modern Europe came to know (Tannery 1886 and Sesiano 1998b).

B	*K*		*K*	*B*
K	*Q*		*Q*	*K*
		X		
K	*Q*		*Q*	*K*
B	*K*		*K*	*B*

Figure 8

We mentioned above the possible connexion between magic squares and chess. From the earliest times, we find that various methods for constructing simple and pandiagonal squares of odd or evenly-even order made use of chess moves: mainly that of the knight (*faras* = horse), that is, a non-diagonal move of two cells in any direction and one cell perpendicularly, but also the move of the queen (*firzān*), that is, a diagonal move to any adjacent cell. Then to complete certain squares of evenly-even order, the complement to $n^2 + 1$ of each number written is placed in the corresponding cell of the bishop (*fīl* = elephant), that is, a diagonal move of two cells in any direction. All these moves, which we shall repeatedly meet, are indicated in Fig. 8: around a cell marked *X* appear all possible knight's moves (*K*), queen's moves (*Q*) and bishop's moves (*B*). The purpose of what follows is to describe some of the methods involving such moves, but above all the knight's move, in the construction of magic squares for odd and evenly-even orders.

2. Odd-order magic squares

- First method

In a blank square of the considered order, start with 1 in the centre cell of the top line. Then proceed downwards from one column to the next using the knight's move. When any side of the square is reached, continue the movement on the opposite side (to determine which cell comes next, imagine the square repeated on the plane). Continue until n numbers have been placed. At this point, no further such move is possible since the next cell is occupied. Staying in the same column, count, whatever the value of n, four cells down; this will be the starting point for the next sequence of n numbers. Repeat these steps until the whole square is complete. This procedure will produce a magic square for any odd order (Fig. 9–12).

- Second method

Place the first n numbers as before but, for the "break-move", count only one cell down. This too will produce a magic square for any odd order (Fig. 9 and 13–15).

Remarks

(1) These two methods are found in many Arabic manuscripts from various periods. We know that the first method reached the Byzantine Empire because it occurs in the above-mentioned treatise by Manuel Moschopoulos.

8	1	6
3	5	7
4	9	2

Figure 9

10	18	1	14	22
4	12	25	8	16
23	6	19	2	15
17	5	13	21	9
11	24	7	20	3

Figure 10

38	14	32	1	26	44	20
5	23	48	17	42	11	29
21	39	8	33	2	27	45
30	6	24	49	18	36	12
46	15	40	9	34	3	28
13	31	7	25	43	19	37
22	47	16	41	10	35	4

Figure 11

(2) Several variants of these two methods are also found in manuscripts. Thus the initial cell is often at the corner and the break-move between two sequences of n numbers one cell back or four cells vertically or horizontally away. Obviously the authors restricted themselves in this case to constructing the first two squares ($n = 5$ and $n = 7$); they are both magic, and even pandiagonal, but the next square, of order $n = 9$, will not be magic – nor will, generally, any square constructed in this last way when n is divisible by 3.

3. Evenly-even squares

• First method

Consider the two squares of order 4 represented in Fig. 16 and 17 (which are both pandiagonal). The underlying construction principle is the same: start with 1 in the top line,

26	58	18	50	1	42	74	34	66
6	38	79	30	71	22	63	14	46
67	27	59	10	51	2	43	75	35
47	7	39	80	31	72	23	55	15
36	68	19	60	11	52	3	44	76
16	48	8	40	81	32	64	24	56
77	28	69	20	61	12	53	4	45
57	17	49	9	41	73	33	65	25
37	78	29	70	21	62	13	54	5

Figure 12

23	12	1	20	9
4	18	7	21	15
10	24	13	2	16
11	5	19	8	22
17	6	25	14	3

Figure 13

46	31	16	1	42	27	12
5	39	24	9	43	35	20
13	47	32	17	2	36	28
21	6	40	25	10	44	29
22	14	48	33	18	3	37
30	15	7	41	26	11	45
38	23	8	49	34	19	4

Figure 14

either in the corner or in a middle cell; place 2 by a knight's move in the next line, 3 in the third line by a queen's move, 4 in the last line by a knight's move again; then do the same for the numbers 8 to 5, starting from the symmetrically located cell in the top line. The configuration of cells thus filled is symmetrical relatively to the vertical axis, and to each occupied cell can be associated one, and only one, empty cell a bishop's move away

77	58	39	20	1	72	53	34	15
6	68	49	30	11	73	63	44	25
16	78	59	40	21	2	64	54	35
26	7	69	50	31	12	74	55	45
36	17	79	60	41	22	3	65	46
37	27	8	70	51	32	13	75	56
47	28	18	80	61	42	23	4	66
57	38	19	9	71	52	33	14	76
67	48	29	10	81	62	43	24	5

Figure 15

14	1	8	*11*
7	*12*	*13*	2
9	6	3	*16*
4	*15*	*10*	5

Figure 16

1	*14*	*11*	8
12	7	2	*13*
6	*9*	*16*	3
15	4	5	*10*

Figure 17

	•	•	
•			•
	•	•	
•			•

Figure 18

(Fig. 18 and 19). The quantity to attribute to an empty cell is determined by subtracting from $n^2 + 1 = 17$ the number found in its associate cell. Thus 16 is associated with 1, 15 with 2, and so forth.

This method can be generalized. We divide a blank square of order $n = 4k$ into subsquares of order 4. As before, we begin in the top line, where we choose two cells in each subsquare, thus $\frac{n}{2}$ cells on the whole, but all either corner cells or middle cells

•			•
	•	•	
•			•
	•	•	

Figure 19

	1	9			24	32	
10			2	31			23
	11	30			3	22	
29			12	21			4
	28	20			13	5	
19			27	6			14
	18	7			26	15	
8			17	16			25

Figure 20

1			13	25			48	60			72
	14	2			47	26			71	59	
15			46	3			70	27			58
	45	16			69	4			57	28	
44			68	17			56	5			29
	67	43			55	18			30	6	
66			54	42			31	19			7
	53	65			32	41			8	20	
52			33	64			9	40			21
	34	51			10	63			22	39	
35			11	50			23	62			38
	12	36			24	49			37	61	

Figure 21

(Fig. 20–21). Half of them will serve as starting points for placing sequences of n consecutive numbers in increasing order beginning with 1, the other half for sequences of decreasing numbers beginning with $\frac{n^2}{2}$. The direction of the first knight's move is easily determined: it must always fall within the subsquare where the initial cell of the sequence

35	1	9	*54*	*43*	24	32	*62*
10	*53*	*36*	2	31	*61*	*44*	23
56	11	30	*64*	*33*	3	22	*41*
29	*63*	*55*	12	21	*42*	*34*	4
58	28	20	*47*	*50*	13	5	*39*
19	*48*	*57*	27	6	*40*	*49*	14
45	18	7	*37*	*60*	26	15	*52*
8	*38*	*46*	17	16	*51*	*59*	25

Figure 22

1	*99*	*130*	13	25	*75*	*142*	48	60	*87*	*118*	72
129	14	2	*100*	*141*	47	26	*76*	*117*	71	59	*88*
15	*132*	*144*	46	3	*97*	*120*	70	27	*73*	*85*	58
143	45	16	*131*	*119*	69	4	*98*	*86*	57	28	*74*
44	*91*	*79*	68	17	*114*	*103*	56	5	*138*	*126*	29
80	67	43	*92*	*104*	55	18	*113*	*125*	30	6	*137*
66	*77*	*101*	54	42	*89*	*128*	31	19	*116*	*140*	7
102	53	65	*78*	*127*	32	41	*90*	*139*	8	20	*115*
52	*134*	*110*	33	64	*122*	*95*	9	40	*107*	*83*	21
109	34	51	*133*	*96*	10	63	*121*	*84*	22	39	*108*
35	*112*	*93*	11	50	*136*	*81*	23	62	*124*	*105*	38
94	12	36	*111*	*82*	24	49	*135*	*106*	37	61	*123*

Figure 23

is located. Then, as for the square of order 4, we progress with knight's moves from one line to the next, except where the movement is interrupted by the side of the main square, whereupon we make one queen's move (which will be towards the last column if we started in a corner cell, away from it otherwise) and then resume with knight's moves. When half of the cells are occupied, we fill the remaining cells by taking the complement to $n^2 + 1$ of each number already placed and putting it in the corresponding bishop's cell within the same subsquare. (See Fig. 22 and Fig. 23, where the complements are to 65 and 145 respectively).

118	1	48	99	87	72	25	142	130	13	60	75
47	100	117	2	26	141	88	71	59	76	129	14
97	46	27	144	120	3	58	73	85	70	15	132
28	143	98	45	57	74	119	4	16	131	86	69
126	29	56	91	79	44	17	114	138	5	68	103
55	92	125	30	18	113	80	43	67	104	137	6
89	54	19	116	128	31	66	101	77	42	7	140
20	115	90	53	65	102	127	32	8	139	78	41
134	21	64	83	107	52	9	122	110	33	40	95
63	84	133	22	10	121	108	51	39	96	109	34
81	62	11	124	136	23	38	93	105	50	35	112
12	123	82	61	37	94	135	24	36	111	106	49

Figure 24

We may choose, as starting points for conjugated pairs of increasing and decreasing sequences (that is, two sequences of which the first terms add up to $\frac{n^2}{2} + 1$), symmetrically located cells (as seen in Fig. 20 with 1, 32 and 9, 24; or in Fig. 21 with 1, 72; 13, 60; 25, 48). But this is not necessary; thus, in Fig. 24, the symmetrically located cells of the top line are occupied by 1 and 60, 48 and 13, 72 and 25. Furthermore, we may start such constructions from any line (Fig. 25). Whatever the initial situation, from then on the same rules apply as before.

This method is described in the smaller of the two anonymous 11th-century treatises (Sesiano 1996a). Its author also mentions a means for checking, before we write in the complements, that the sequences have been placed correctly: first, the sum appearing in all the lines must be the same; secondly, any column must contain the same sum as the next column but one within the same column of subsquares (as seen in Fig. 20–21). This, incidentally, explains why the completed square will be magic: the sum of the complements for *any* line is the same as for any other, while the sum of the complements for *either of any two* conjugate columns is the same as for the other; filling in the bishop's cells will therefore complete the amount needed for the magic sum. Indeed, if in a line or a column we have initially placed $\frac{n}{2}$ numbers α_i making the sum $\sum \alpha_i$, the conjugate line or column will receive their complements $(n^2 + 1) - \alpha_i$. Since the cells already filled contain $\frac{n}{2}$ numbers β_i with the sum $\sum \alpha_i = \sum \beta_i$, the sum in the conjugate line or column will be

$$\sum \beta_i + \frac{n}{2}(n^2 + 1) - \sum \alpha_i = \frac{n}{2}(n^2 + 1)$$

that is, the magic sum. Thirdly, the case of the diagonals is banal: since they are composed of pairs of complements, they must contain $\frac{n}{2}$ times the amount $n^2 + 1$.

89	46	27	116	128	3	58	101	77	70	15	140
28	115	90	45	57	102	127	4	16	139	78	69
118	29	56	99	87	44	17	142	130	5	68	75
55	100	117	30	18	141	88	43	67	76	129	6
81	54	19	124	136	31	66	93	105	42	7	112
20	123	82	53	65	94	135	32	8	111	106	41
126	21	64	91	79	52	9	114	138	33	40	103
63	92	125	22	10	113	80	51	39	104	137	34
97	62	11	144	120	23	38	73	85	50	35	132
12	143	98	61	37	74	119	24	36	131	86	49
134	1	48	83	107	72	25	122	110	13	60	95
47	84	133	2	26	121	108	71	59	96	109	14

Figure 25

• Second method

Starting with the sequences beginning with 1 and $\frac{n^2}{2}$ taken in increasing and decreasing order respectively, we proceed to the end of the first pair of lines using the knight's move, then advance to the following pair of lines with a queen's move and return in the opposite direction (Fig. 26). Continuing thus, we place the first half of the numbers, and then fill the empty bishop's cells within each subsquare of order 4 with their complements (Fig. 27).

This method, found in Shabrāmallisī's treatise, is also described earlier, in the larger of the two anonymous 11th-century works (Sesiano 1996b, pp. 44–45 & 195–196). Its author considers at length various positions the initial cell may take.

The result is indeed a magic square for, here too, the first part of the procedure has left each line with pairs of cells containing the sum $\frac{n^2}{2}+1$ (thus $\frac{n}{4}(\frac{n^2}{2}+1)$ for the whole line). Conjugate columns also contain a same sum. The complements will therefore complete the magic sum. Finally, each of the two main diagonals contains pairs of complements. Note that in this case the square is even pandiagonal, a property not noted by the author. The same holds for the squares constructed by the next method.

• Third method

"Knight's moves in four cycles" is how Muḥammad Shabrāmallisī denominates this method; for here we place all four sequences of $\frac{n^2}{4}$ numbers using the knight's move. We begin by writing the first $\frac{n^2}{4}$ numbers, thus from 1 to 4 in Fig. 28, from 1 to 16 in Fig. 29, from 1 to 36 in Fig. 30. To do this, we use the knight's move to place $\frac{n}{2}$ numbers within the first two lines; we then repeat this movement along the next pair of lines but this time starting one cell away from the side. When this is done for the whole square, we proceed

1			68	3			70	5			72
	67	2			69	4			71	6	
66			11	64			9	62			7
	12	65			10	63			8	61	
13			56	15			58	17			60
	55	14			57	16			59	18	
54			23	52			21	50			19
	24	53			22	51			20	49	
25			44	27			46	29			48
	43	26			45	28			47	30	
42			35	40			33	38			31
	36	41			34	39			32	37	

Figure 26

1	*134*	*79*	68	3	*136*	*81*	70	5	*138*	*83*	72
80	67	2	*133*	*82*	69	4	*135*	*84*	71	6	*137*
66	*77*	*144*	11	64	*75*	*142*	9	62	*73*	*140*	7
143	12	65	*78*	*141*	10	63	*76*	*139*	8	61	*74*
13	*122*	*91*	56	15	*124*	*93*	58	17	*126*	*95*	60
92	55	14	*121*	*94*	57	16	*123*	*96*	59	18	*125*
54	*89*	*132*	23	52	*87*	*130*	21	50	*85*	*128*	19
131	24	53	*90*	*129*	22	51	*88*	*127*	20	49	*86*
25	*110*	*103*	44	27	*112*	*105*	46	29	*114*	*107*	48
104	43	26	*109*	*106*	45	28	*111*	*108*	47	30	*113*
42	*101*	*120*	35	40	*99*	*118*	33	38	*97*	*116*	31
119	36	41	*102*	*117*	34	39	*100*	*115*	32	37	*98*

Figure 27

with the second cycle: we move sideways from the cell just reached and return, using the knight's move, along the same pair of lines; and, because we started one cell away from the side in the last line, we shall start, in the pair of lines above, in the first cell.

1	**8**	*11*	*14*
12	*13*	**2**	**7**
6	**3**	*16*	*9*
15	*10*	**5**	**4**

Figure 28

1	**32**	*47*	*50*	**3**	**30**	*45*	*52*
48	*49*	**2**	**31**	*46*	*51*	**4**	**29**
28	**5**	*54*	*43*	**26**	**7**	*56*	*41*
53	*44*	**27**	**6**	*55*	*42*	**25**	**8**
9	**24**	*39*	*58*	**11**	**22**	*37*	*60*
40	*57*	**10**	**23**	*38*	*59*	**12**	**21**
20	**13**	*62*	*35*	**18**	**15**	*64*	*33*
61	*36*	**19**	**14**	*63*	*34*	**17**	**16**

Figure 29

1	**72**	*107*	*110*	**3**	**70**	*105*	*112*	**5**	**68**	*103*	*114*
108	*109*	**2**	**71**	*106*	*111*	**4**	**69**	*104*	*113*	**6**	**67**
66	**7**	*116*	*101*	**64**	**9**	*118*	*99*	**62**	**11**	*120*	*97*
115	*102*	**65**	**8**	*117*	*100*	**63**	**10**	*119*	*98*	**61**	**12**
13	**60**	*95*	*122*	**15**	**58**	*93*	*124*	**17**	**56**	*91*	*126*
96	*121*	**14**	**59**	*94*	*123*	**16**	**57**	*92*	*125*	**18**	**55**
54	**19**	*128*	*89*	**52**	**21**	*130*	*87*	**50**	**23**	*132*	*85*
127	*90*	**53**	**20**	*129*	*88*	**51**	**22**	*131*	*86*	**49**	**24**
25	**48**	*83*	*134*	**27**	**46**	*81*	*136*	**29**	**44**	*79*	*138*
84	*133*	**26**	**47**	*82*	*135*	**28**	**45**	*80*	*137*	**30**	**43**
42	**31**	*140*	*77*	**40**	**33**	*142*	*75*	**38**	**35**	*144*	*73*
139	*78*	**41**	**32**	*141*	*76*	**39**	**34**	*143*	*74*	**37**	**36**

Figure 30

1	**4**	*14*	*15*
13	*16*	**2**	**3**
8	**5**	*11*	*10*
12	*9*	**7**	**6**

Figure 31

Thus we proceed with this second cycle just as for the first $\frac{n^2}{4}$ numbers but reversing the two directions of the movement. After arriving next to our initial cell, we put the subsequent number, the first of the third cycle, in the other end of the broken diagonal, that is, above the last cell (9 in Fig. 28, 33 in Fig. 29, 73 in Fig. 30); we then continue with knight's moves in the same manner as before, advancing to the left and upwards. After arriving below our initial cell, we again move, for the last cycle, sideways to the next cell and proceed to the right and down until the last number is placed.

The reason that the lines and columns produce the magic sum is the same as in the preceding method. Indeed, the $\frac{n^2}{2}$ elements placed after the first two cycles differ from the previous case in arrangement only: half the columns, namely those formerly of odd indices, are the same, while the columns of even indices $2i$ are now those with indices $n - 2i + 2$; thus the elements in the lines will be the same as before while the conjugate columns will form the same pairs. This will, however, mean that we end up with different numbers in the two diagonals, though still forming, in the end, pairs of complements.

- Fourth method

The same author describes another method, which he calls "knight's moves in two cycles". We start as above but, instead of advancing to the next pair of lines, return along the same ones after making one move sideways from the cell last reached. The move to each subsequent pair of lines is as in the previous method. This brings us, in the last pair of lines, to the number $\frac{n^2}{2}$ (8 in Fig. 31, 72 in Fig. 32), whereupon we move to the corresponding lower queen's cell and resume the movement in the opposite direction. The construction will end in the queen's cell of 1.

After half the numbers have been placed, all columns produce the same sum and, if we consider successive pairs of *adjacent* columns, each will receive the complements of its neighbour. Equal sums are also found in pairs of adjacent lines, each of which will be completed with the complements of the other. Finally, the diagonals will contain in the end pairs of complements, but this time in adjacent cells.

- Fifth method

This method is explained in the larger of the two anonymous 11th-century treatises (Sesiano 1996b, pp. 66–67 & 177–179). Starting with 1 and n^2 at either end of (say) the top line, we use the knight's move to place two (increasing and decreasing) sequences of $\frac{n}{2}$ numbers within the first two lines. We then repeat the movement along the next pair of lines starting, as we did in the third method, one cell away from the side. We continue

1	12	*134*	*143*	3	10	*136*	*141*	5	8	*138*	*139*
133	*144*	2	11	*135*	*142*	4	9	*137*	*140*	6	7
24	13	*131*	*122*	22	15	*129*	*124*	20	17	*127*	*126*
132	*121*	23	14	*130*	*123*	21	16	*128*	*125*	19	18
25	36	*110*	*119*	27	34	*112*	*117*	29	32	*114*	*115*
109	*120*	26	35	*111*	*118*	28	33	*113*	*116*	30	31
48	37	*107*	*98*	46	39	*105*	*100*	44	41	*103*	*102*
108	*97*	47	38	*106*	*99*	45	40	*104*	*101*	43	42
49	60	*86*	*95*	51	58	*88*	*93*	53	56	*90*	*91*
85	*96*	50	59	*87*	*94*	52	57	*89*	*92*	54	55
72	61	*83*	*74*	70	63	*81*	*76*	68	65	*79*	*78*
84	*73*	71	62	*82*	*75*	69	64	*80*	*77*	67	66

Figure 32

1			*16*
	15	2	
	3	*14*	
13			4

Figure 33

in the same way, for each group of four lines, to the bottom, thus placing the first and last $\frac{n^2}{4}$ numbers (1 to 4 and 16 to 13 in Fig. 33, 1 to 36 and 144 to 109 in Fig. 34).

The remaining numbers are then placed, not by using the knight's move, but progressing along each broken diagonal as indicated by arrows in Fig. 35–36. First, we return to the increasing sequence (with, in the examples, 5 and 37 respectively) in the column where we left off, but starting now in the third cell from the top, and fill in the empty cells of the corresponding broken diagonal. We do the same for the next broken diagonals, starting each time two cells below; the first cell actually filled is thus alternately in the first and second column. This brings us back to the top of the square. In this way we shall place the sequences 5, 6 and 7, 8 in Fig. 35, and 37 to 42, 43 to 48, 49 to 54, 55 to 60, 61 to 66, and 67 to 72 in Fig. 36. Lastly, we write the remaining $\frac{n^2}{4}$ numbers along the corresponding ascending diagonals (starting with 12 in Fig. 35 and 108 in Fig. 36).

1			*140*	3			*142*	5			*144*
	139	2			*141*	4			*143*	6	
	7	*134*			9	*136*			11	*138*	
133			8	*135*			10	*137*			12
13			*128*	15			*130*	17			*132*
	127	14			*129*	16			*131*	18	
	19	*122*			21	*124*			23	*126*	
121			20	*123*			22	*125*			24
25			*116*	27			*118*	29			*120*
	115	26			*117*	28			*119*	30	
	31	*110*			33	*112*			35	*114*	
109			32	*111*			34	*113*			36

Figure 34

1	6	*11*	*16*
8	*15*	2	*9*
12	3	*14*	5
13	*10*	7	4

Figure 35

This method produces a magic square for the following reasons. The lines produce the required amount since each line contains pairs of complements (in cells which are symmetrically located). The columns contain $\frac{n}{4}$ arithmetical progressions with $\frac{n}{4}$ terms each, namely, if i designates any of the numbers from 1 to n and t takes the natural values $t = 1, \ldots, \frac{n}{4}$,

$$i + (t-1)n$$

$$\left(\frac{n^2}{2} - i + 1\right) - (t-1)n$$

$$\left(\frac{n^2}{2} + n - i + 1\right) + (t-1)n$$

$$(n^2 - n + i) - (t-1)n,$$

1	66	107	140	3	52	93	142	5	38	79	144
72	139	2	101	58	141	4	87	44	143	6	73
108	7	134	65	94	9	136	51	80	11	138	37
133	102	71	8	135	88	57	10	137	74	43	12
13	42	95	128	15	64	81	130	17	50	103	132
48	127	14	89	70	129	16	75	56	131	18	97
96	19	122	41	82	21	124	63	104	23	126	49
121	90	47	20	123	76	69	22	125	98	55	24
25	54	83	116	27	40	105	118	29	62	91	120
60	115	26	77	46	117	28	99	68	119	30	85
84	31	110	53	106	33	112	39	92	35	114	61
109	78	59	32	111	100	45	34	113	86	67	36

Figure 36

the sum of which for i constant gives the magic sum. Finally, consider the descending main diagonal. It is occupied by numbers belonging to the arithmetical progressions

$$1 + s\left(\frac{n}{2} + 1\right)$$
$$n^2 - s'\left(\frac{n}{2} - 1\right)$$

with $s = 0, 1, \ldots, \frac{n}{2} - 1$ and $s' = 1, 2, \ldots, \frac{n}{2}$. The sum of the n numbers belonging to these two progressions will therefore be

$$1 \cdot \frac{n}{2} + \frac{n}{2}\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} + 1\right) + n^2 \cdot \frac{n}{2} - \frac{n}{2}\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} - 1\right) = \frac{n}{2}(n^2 + 1).$$

The other main diagonal, since it contains their complements, will produce the magic sum as well. Figure 37 will help familiarize the reader with this last method.

We have provided all these methods with a justification for the magic property. Most Arabic texts do not, however, explain why a square obtained by a particular method happened to be magic. (Exceptions are found in some works by mathematicians, such as those by Abū'l-Wafā' and Ibn al-Haytham.) Clearly then, empiricism may have led to the discovery of some of the above methods thought at the time to be general (see the second remark concluding Sect. 2). Elsewhere though, there is quite definite evidence pointing to a theoretical foundation (if only found *a posteriori*), as in the first method

1	120	191	**250**	**3**	102	173	**252**	**5**	84	155	**254**	**7**	66	137	**256**
128	**249**	**2**	183	110	**251**	**4**	165	92	**253**	**6**	147	74	**255**	**8**	129
192	**9**	**242**	119	174	**11**	**244**	101	156	**13**	**246**	83	138	**15**	**248**	65
241	184	127	**10**	**243**	166	109	**12**	**245**	148	91	**14**	**247**	130	73	**16**
17	72	175	**234**	**19**	118	157	**236**	**21**	100	139	**238**	**23**	82	185	**240**
80	**233**	**18**	167	126	**235**	**20**	149	108	**237**	**22**	131	90	**239**	**24**	177
176	**25**	**226**	71	158	**27**	**228**	117	140	**29**	**230**	99	186	**31**	**232**	81
225	168	79	**26**	**227**	150	125	**28**	**229**	132	107	**30**	**231**	178	89	**32**
33	88	159	**218**	**35**	70	141	**220**	**37**	116	187	**222**	**39**	98	169	**224**
96	**217**	**34**	151	78	**219**	**36**	133	124	**221**	**38**	179	106	**223**	**40**	161
160	**41**	**210**	87	142	**43**	**212**	69	188	**45**	**214**	115	170	**47**	**216**	97
209	152	95	**42**	**211**	134	77	**44**	**213**	180	123	**46**	**215**	162	105	**48**
49	104	143	**202**	**51**	86	189	**204**	**53**	68	171	**206**	**55**	114	153	**208**
112	**201**	**50**	135	94	**203**	**52**	181	76	**205**	**54**	163	122	**207**	**56**	145
144	**57**	**194**	103	190	**59**	**196**	85	172	**61**	**198**	67	154	**63**	**200**	113
193	136	111	**58**	**195**	182	93	**60**	**197**	164	75	**62**	**199**	146	121	**64**

Figure 37

presented for evenly-even squares. Whether theoretically or empirically reached, these discoveries are the result of an amazing amount of study which cannot but compel our admiration.

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